A SOLUTION OF THE DIFFUSION EQUATION WITH NONLINEAR RIGHT-HAND SIDE

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The division of a rectangular pile of subsoil water between two channels with different water levels is considered. Evaporation is allowed for as a function of the depth of the subsoil stream. The problem reduces to solution of a nonlinear integral equation and is solved approximately by a method developed from the method of successive replacement of stationary states. A numerical example is given.

1. We seek the solution of the diffusion equation

$$\frac{\partial h}{\partial t} = a^2 \frac{\partial^2 h}{\partial x^2} + f(h) \tag{1.1}$$

where $f(h)$ is the nonlinear function

$$f(h) = \begin{cases} 0 & (h < h_0) \\ -bh + cH & (h > h_0) \end{cases} \qquad (b \geqslant 0) \tag{1.2}$$

in the interval $0 < x < L$ with the initial condition

$$h(x, 0) = \varphi(x) \tag{1.3}$$

and the boundary conditions

$$h(0, t) = H_1, \qquad h(L, t) = H_2 \tag{1.4}$$

This problem is of the type discussed in [1–5].

The problem of integrating Eqs. (1.1) and (1.2) under conditions (1.3) and (1.4) arises when considering the division of a pile of subsoil water of the initial shape (1.3) between two channels spaced a distance L apart, under the assumption that at the initial instant the water levels in the channels suddenly become equal to H_1 and H_2 and under the additional assumptions that evaporation occurs when $h > h_0$ and does not occur when $h < h_0$. The degree of evaporation $f(h)$ is assumed to be either a linear function of the subsoil water level h ($b > 0$, $c = b$, $H = h_0$) or constant ($b = 0$, $cH < 0$).

In (1.1) $a^2 = kH_*/\sigma$, where k is a filtering factor, H_* is the mean depth of the subsoil stream, and σ is the deficiency of saturation or water yield.

Consider the case in which the level of the subsoil water is constant at the initial instant:

$$h(x, 0) = H_0 \tag{1.5}$$

where it will be assumed that $H_1 < h_0 \leq H_0 \leq H_2$. Problem (1.1), (1.2), (1.4), (1.5) then reduces to solution of the equations

$$\frac{\partial h_1}{\partial t} = a^2 \frac{\partial^2 h_1}{\partial x^2}, \quad h_1(0, t) = H_1, \quad h_1(\chi(t), t) = h_0 \qquad (0 < x < \chi(t)) \tag{1.6}$$

and

$$\frac{\partial h_2}{\partial t} = a^2 \frac{\partial^2 h_2}{\partial x^2} - bh_2 + cH \tag{1.7}$$

$$h_2(x, 0) = H_0, \quad h_2(L, t) = H_2, \quad h_2[\chi(t), t] = h_0 \qquad (\chi(t) < x < L)$$

Here, $x = \chi(t)$ is a boundary, not known in advance, on which

$$h[\chi(t), t] = h_0, \quad \frac{\partial h_1[\chi(t), t]}{\partial x} = \frac{\partial h_2[\chi(t), t]}{\partial x} \tag{1.8}$$

The last condition implies the continuity of the flow in the cross section $x = \chi(t)$; after assuming that $\chi(t)$ is such that $\chi(0) = 0$, it is well known that the functions $h_1(x, t)$ and $h_2(x, t)$ may be found from (1.6) and (1.7), while $\chi(t)$ is given by the second of Eqs. (1.8).

Consider the asymptotic form of the solution of our problem close to $t = 0$, $x = 0$; we have

$$h_1(x, t) = h_2(x, t) = H_1 + (H_0 - H_1)\,\Phi\left(\frac{x}{2a\sqrt{t}}\right)$$

$$\left(\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-s^2)\,ds\right) \tag{1.9}$$

Eqs. (1.6) and (1.7) are then satisfied up to higher-order terms, while initial condition (1.5), the first of the boundary conditions (1.4), and the second of conditions (1.8) are also satisfied. On satisfying the first boundary condition of (1.8), an equation is obtained for the curve $x = \chi(t)$ in the neighborhood of $t = 0$, $x = 0$:

$$x = \chi(t) = D\sqrt{t} \tag{1.10}$$

where D is the unique root of the equation

$$\Phi\left(\frac{D}{2a}\right) = \frac{h_0 - H_1}{H_0 - H_1} \tag{1.11}$$

Take the case $b > 0$, $c = b$, $H = h_0$. Substituting

$$v = (h - h_0)e^{bt} \tag{1.12}$$

problem (1.7) reduces to solving the heat-conduction equation

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} \tag{1.13}$$

$$v(x, 0) = H_0 - h_0 \tag{1.14}$$

$$v(L, t) = (H_2 - h_0)e^{bt}, \quad v[\chi(t), t] = 0, \quad (0 < \chi(t) < L) \tag{1.15}$$

The second condition of (1.8) here transforms to

$$\frac{\partial h_1[\chi(t), t]}{\partial x} = \exp(-bt)\,\frac{\partial v[\chi(t), t]}{\partial x} \tag{1.16}$$

Instead of $h_1(x, t)$, we consider the function $h_1(x, t) - h_0$. The initial condition (1.6) then becomes

$$h_1[\chi(t), t] - h_0 = 0 \tag{1.17}$$

A problem similar to that described by Eqs. (1.6) and (1.13) under initial condition (1.14) and boundary conditions (1.6), (1.17) and (1.15), (1.16) was considered in [2].

It can be seen that $h_1(x, t)$ is given by

$$h_1(x, t) = h_0 + \frac{a}{2\sqrt{\pi}}\int_0^t \frac{v_1(\tau)}{\sqrt{t-\tau}}\left[\exp\left\{-\frac{[x - \chi(\tau)]^2}{4a^2(t-\tau)}\right\}\right.$$

$$\left.- \exp\left\{-\frac{[x + \chi(\tau)]^2}{4a^2(t-\tau)}\right\}\right] d\tau + (H_1 - h_0)\left[1 - \Phi\left(\frac{x}{2a\sqrt{t}}\right)\right]$$

$$\left(\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-s^2)\,ds\right) \tag{1.18}$$

The following expression is obtained for $v(x, t)$:

$$v(x,t) = -\frac{a}{2\sqrt{\pi}}\int_0^t \frac{v_2(\tau)}{\sqrt{t-\tau}}\left[\exp\left\{-\frac{[x-\chi(\tau)]^2}{4a^2(t-\tau)}\right\}\right.$$
$$\left.-\exp\left\{-\frac{[x+\chi(\tau)-2L]^2}{4a^2(t-\tau)}\right\}\right]d\tau$$
$$+\frac{(H_0-h_0)}{2a\sqrt{\pi t}}\int_0^L\left[\exp\left\{-\frac{(x-\xi)^2}{4a^2t}\right\}-\exp\left\{-\frac{(x+\xi-2L)^2}{4a^2t}\right\}\right]d\xi \tag{1.19}$$
$$-\frac{(H_2-h_0)}{2a\sqrt{\pi}}\int_0^t\frac{x-L}{(t-\tau)^{3/2}}\exp\left\{b\tau-\frac{(x-L)^2}{4a^2(t-\tau)}\right\}d\tau$$

Here,

$$v_1(t) = \left.\frac{\partial(h_1-h_0)}{\partial x}\right|_{x=\chi(t)-0},\qquad v_2(t) = \left.\frac{\partial v}{\partial x}\right|_{x=\chi(t)+0} \tag{1.20}$$

Using the properties of the heat potential of a double layer [6], the following integral equations for the functions $v_1(t)$, $v_2(t)$, and $\chi(t)$ are obtained from (1.18)–(1.20):

$$v_1(t) = \int_0^t K_1(t,\tau)\,v_1(\tau)\,d\tau + \eta(t),\qquad v_2(t) = -\int_0^t K_2(t,\tau)\,v_2(\tau) + \gamma(t) \tag{1.21}$$

Here,

$$K_i(t,\tau) = -\frac{1}{2a\sqrt{\pi}}\frac{1}{(t-\tau)^{3/2}}\left\{[\chi(t)-\chi(\tau)]\exp\left\{-\frac{[\chi(t)-\chi(\tau)]^2}{4a^2(t-\tau)}\right\}\right. \tag{1.22}$$
$$\left.-[\chi(t)+\chi(\tau)-2(i-1)L]\exp\left\{-\frac{[\chi(t)+\chi(\tau)-2(i-1)L]^2}{4a^2(t-\tau)}\right\}\right\}\qquad (i=1,2)$$
$$\eta(t) = -\frac{2(H_1-h_0)}{a\sqrt{\pi t}}\exp\left\{-\frac{\chi^2(t)}{4a^2t}\right\} \tag{1.23}$$
$$\gamma(t) = \frac{2b(H_2-h_0)}{a\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{t-\tau}}\exp\left\{b\tau-\frac{(\chi(t)-L)^2}{4a^2(t-\tau)}\right\}d\tau$$
$$+\frac{2(H_2-H_0)}{a\sqrt{\pi t}}\exp\left[-\frac{(\chi(t)-L)^2}{4a^2t}\right]$$
$$+\frac{(H_0-h_0)}{a\sqrt{\pi t}}\left\{\exp\left[-\frac{\chi^2(t)}{4a^2t}\right]+\exp\left[-\frac{(\chi(t)-2L)^2}{4a^2t}\right]\right\} \tag{1.24}$$

The solutions of the Volterra integral equations of the second kind (1.21) are [7]

$$v_1(t) = \xi(t) + \int_0^t \xi(\tau)\,S_1(t,\tau)\,d\tau,\qquad v_2(t) = \mu(t) + \int_0^t \mu(\tau)\,S_2(t,\tau)\,d\tau \tag{1.25}$$

Here,

$$\xi(t) = \eta(t) + \int_0^t K_1(t,\tau)\,\eta(\tau)\,d\tau,\qquad \mu(t) = \gamma(t) - \int_0^t K_2(t,\tau)\,\gamma(\tau)\,d\tau$$
$$G_1^{(i)}(t,\tau) = \int_\tau^t K_i(t,\sigma)\,K_i(\sigma,\tau)\,d\sigma \tag{1.26}$$
$$G_{m+1}^{(i)}(t,\tau) = \int_\tau^t G_1^{(i)}(t,\sigma)G_m^{(i)}(\sigma,\tau)\,d\sigma,\qquad S_i(t,\tau) = \sum_{m=1}^{\infty} G_m^{(i)}(t,\tau)\qquad (i=1,2)$$

Using (1.16), (1.20), and (1.22)–(1.26), we obtain the following integral equation for $\chi(t)$ (such that $\chi(0) = 0$):

$$\gamma(t) - e^{bt}\eta(t) = \int_0^t [K_2(t,\tau)\,\gamma(\tau) + e^{bt}\eta(\tau)\,K_1(t,\tau)]\,d\tau$$
$$+\int_0^t [\xi(\tau)\,S_1(t,\tau)\,e^{bt} - \mu(\tau)\,S_2(t,\tau)]\,d\tau \tag{1.27}$$

2. Let the initial level of the subsoil water be equal to the water level in the channel when x = L, $H_0 = H_2$. The above problem may be solved approximately, and the function $\chi(t)$ obtained, by using a development of the method of successively replacing the stationary states [8.9].

We assume that Eq. (1.1), with $f(h)$ on its right-hand side given by (1.2), is only satisfied in the integral sense, i.e., after it has been integrated over the entire range $0 \le x \le L$ of the independent variable x:

$$\int_0^L \frac{\partial h}{\partial t}\,dx = \int_0^L a^2 \frac{\partial^2 h}{\partial x^2}\,dx + \int_0^L f(h)\,dx \tag{2.1}$$

The functions $h_1(x, t)$ and $h_2(x, t)$, satisfying, respectively, boundary conditions (1.7), (1.8) and (1.11), (1.12), will be defined with $b \neq 0$ by the expressions

$$h_1(x, t) = h_0 + (h_0 - H_1)\frac{[x - l(t)]}{l(t)} + C(t)\,x\,[x - l(t)] \qquad (0 \leqslant x \leqslant l(t)) \tag{2.2}$$

$$h_2(x, t) = \frac{cH}{b} + \left(h_0 - \frac{cH}{b}\right)\frac{\operatorname{sh}\omega(L - x)}{\operatorname{sh}\omega[L - l(t)]}$$
$$-\left(H_2 - \frac{cH}{b}\right)\frac{\operatorname{sh}\omega[l(t) - x]}{\operatorname{sh}\omega[L - l(t)]} \qquad \left(\omega = \frac{\sqrt{b}}{a},\ l(t) \leqslant x \leqslant L\right) \tag{2.3}$$

Two unknown functions of t, C(t) and $l(t)$, appear in (2.2) and (2.3). Here, $l(t)$ is the approximate value of the function $\chi(t)$.

The problem now reduces to finding the functions C(t) and $l(t)$.

To find C(t), Eq. (1.14) will be used:

$$\frac{\partial h_1[l(t), t]}{\partial x} = \frac{\partial h_2[l(t), t]}{\partial x} \tag{2.4}$$

Hence

$$C = \frac{1}{l}\left\{\frac{H_1 - h_0}{l} - \frac{\omega}{\operatorname{sh}\omega(L - l)}\left[\left(h_0 - \frac{cH}{b}\right)\operatorname{ch}\omega(L - l) - \left(H_2 - \frac{cH}{b}\right)\right]\right\} \tag{2.5}$$

Using (2.1) in conjunction with (2.2), (2.3), and (2.5), an ordinary linear differential equation can be obtained for $l(t)$, which is integrable in quadratures:

$$t = \int_0^l F(z)\,dz \tag{2.6}$$

$$F(z) = \frac{z}{f(z)}\{\beta \operatorname{sh}^2\omega(L - z) + 2\delta z \operatorname{sh}\omega(L - z)\operatorname{ch}\omega(L - z)$$
$$+ 2\nu z \operatorname{sh}\omega(L - z) + \omega[\alpha + \delta z^2 + (\mu + \nu z^2)\operatorname{ch}\omega(L - z)]\}$$

$$f(z) = 2a^2 \operatorname{sh}\omega(L - z)\ (H_1 - h_0)\operatorname{sh}\omega(L - z) - Az\operatorname{ch}\omega(L - z) + Bz\}$$

Here,

$$A = \omega\left(h_0 - \frac{cH}{b}\right), \quad B = \omega\left(H_2 - \frac{cH}{b}\right), \quad \alpha = \frac{1}{\omega}\left(h_0 + H_2 - 2\frac{cH}{b}\right)$$
$$B = -cH / b + {}^2/_3 h_0 + {}^1/_3 H_1, \qquad \delta = {}^1/_6 A, \qquad \nu = {}^1/_6 B, \qquad \mu = -\alpha \tag{2.7}$$

It can be shown that

$$F(z) \to \infty \quad \text{as} \quad z \to \chi_0$$

where χ_0 is the root of the equation C = 0, and C is given by (2.5).

Integral (2.6) is divergent when $l = \chi_0$; consequently, $t \to \infty$. Putting $l(t) = \chi_0$ and C = 0 in (2.2) and (2.3), we obtain the solution of the stationary problem discussed here. In the neighborhood of $l = 0$, we have

$$l = M\sqrt{t} \tag{2.8}$$

$$M = 2a\sqrt{h_0 - H_1}\left[\frac{1}{3}\left(H_2 + h_0 - 2\frac{cH}{b}\right)\operatorname{sch}^2\frac{\omega L}{2} + \left(-\frac{2}{3}h_0 - \frac{H_1}{3} + \frac{cH}{b}\right)\right]^{-1/2} \tag{2.9}$$

When $b \neq 0$, we have to put $c = b$ and $H = h_0$ in (2.2), (2.3), (2.5)−(2.7), and (2.9), which then give the approximate solution of our problem, whereas when $b = 0$, the solution is obtained by a passage to the limit in these expressions. Let us write down this solution:

$$h_1(x, t) = h_0 + (h_0 - H_1)\frac{x - l}{l} + Cx(x - l) \qquad (0 \leqslant x \leqslant l) \tag{2.10}$$

$$h_2(x, t) = H_2 + \frac{H_2 - h_0}{L - l}(x - L) - \frac{cH}{2a^2}(x - l)(x - L) \qquad (l \leqslant x \leqslant L) \tag{2.11}$$

$$C = \frac{1}{l}\left[\frac{H_2 - h_0}{L - l} - \frac{h_0 - H_1}{l} + \frac{cH}{2a^2}(L - l)\right] \tag{2.12}$$

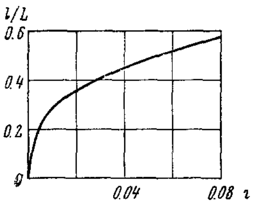
$$f(z) = \frac{z}{(L - z)}\,\frac{\alpha(L - z)^2 + \beta z(2L - z) + \eta(4z - 3L)(L - z)^2}{\zeta z(L - z)^2 + \gamma(L - z) + \delta z} \tag{2.13}$$

$$\alpha = \frac{h_0 + 2H_1 - 3H_2}{6}, \quad \beta = \frac{h_0 - H_2}{6}, \quad \eta = \frac{cHL}{12a^2}, \quad \zeta = cH \tag{2.14}$$

$$\gamma = -2a^2(h_0 - H_1), \qquad \delta = 2a^2(H_2 - h_0)$$

$$M = \frac{2a\sqrt{h_0 - H_1}}{\sqrt{{}^1/_6(3H_2 - 2H_1 - h_0 + {}^3/_2 cHa^{-2}L^2)}} \tag{2.15}$$

The solution of the corresponding stationary problem is again obtained from (2.10)–(2.12) with $l = \chi_0$ and $C = 0$.



Using (2.8) and (2.15), the condition $M > 0$ leads to an inequality imposed on the quantities H_2, H_1, h_0, L, a^2, cH:

$$H_2 > \frac{h_0}{3} + \frac{2}{3}H_1 - \frac{1}{2}\frac{cHL^2}{a^2} \tag{2.16}$$

which is satisfied in turn if

$$H_2 \geqslant h_0 - \frac{1}{2}\frac{cHL^2}{a^2} \tag{2.17}$$

Inequality (2.17) represents the condition for the slope of the tangent at any point of parabola (2.10) to be nonnegative; when condition (2.17) is satisfied, the slope of the tangent at any point of parabola (2.11) is likewise nonnegative, and the inequalities $h_1 \leq h_0$, $h_2 \geq h_0$ will hold for the solution (2.10), (2.11) when $b = 0$.

When $b \neq 0$, it follows from (2.9) that $M > 0$ for any h_0 $(H_2 > h_0 > H_1)$. From (2.2) and (2.3), where we have to put $c = b$ and $H = h_0$, we have $\partial h_1/\partial x > 0$, $\partial h_2/\partial x > 0$, i.e., $h_1 \leq h_0$, $h_2 \geq h_0$.

The approximate solution of our problem has now been obtained.

The accompanying figure gives the dependence of l/L on $\tau = a^2t/L^2$, as given by (2.6). Here, $H_2 = 1.25\ h_0$, $H_1 = 0.5\ h_0$, $bL^2/a^2 = 1.25$, so that $\chi_0/L = 0.672$.

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